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A GENERALIZED MAXIMUM ENTROPY PRINCIPLE

FOR DECISION ANALYSIS

by

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NAVAL POSTGRADUATE SCHOOL
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ABSTRACT

A generalized maximum entropy principle is described for dealing with decision problems involving uncertainty but with some prior knowledge about the probability space corresponding to nature. This knowledge about the probalistic structure is expressed through known bounds on event probabilities and moments, which is incorporated into a nonlinear programming problem. The solution provides a maximum entropy distribution which is then used in treating the decision problem as one involving risk.

An example application is described that involves the selection of oil spill recovery systems for inland harbor regions. Other areas of application are identified and tables of some maximum entropy distributions resulting from a variety of moment constraints are provided.

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An example application is described that involves the selection of oil spill recovery systems for inland harbor regions. Other areas of application are identified and tables of some maximum entropy distributions resulting from a variety of moment constraints are provided.

This paper describes a general maximum entropy principle (GMEP) for dealing with uncertainty within a decision analytic framework. These decision analysis problems are quite common, and in fact are fundamental, in operations research practice where one is trying to make the most out of some limited knowledge and resources. Entropy is a measure of dispersion that is widely used for quantifying uncertainty. Stemming from early work by Shannon [18] in communications theory, it has been applied rigorously in psychology and engineering for quantifying complexity and various indices of workload related to human performance (e.g. see Thomas [22]). Recently it has been used as a performance measure in areas of business and economics [1, 4, 9, 10, 21, 27], behavioral sciences [8, 20], and physical sciences [7, 23]. Mathematical entropy as a criterion for resolving choices among alternatives has received acceptance in location and distribution analysis [12, 26], the design of questionnaires [5], statistical inference [6, 11], and to an extent in search problems [13].

Before we discuss the maximum entropy principle we shall present a basic formulation of a decision analysis problem. The general principle involves deriving a maximum entropy distribution for given knowledge about the set of states of nature. This derivation as stated or alluded to by others seems to require that the given knowledge be limited to that which can be expressed through the moments of an associated random variable. Here we relax this restriction somewhat by allowing for information through bounded event probabilities and moments. A numerical example is given to demonstrate application of the maximum entropy principle in making selections among alternative oil pollution abatement systems. There are a number of problems in systems design and evaluation for which this principle offers a pragmatic approach. Some of the more common areas of operations research are summarized followed by some concluding remarks.

1. THE DECISION PROBLEM.

A decision problem is specified by the triplet (A, Ω, V), where A is a set of alternatives available to the decision maker subject to effects from a set of possible future states of nature Ω to which a set of values V is associated. Both A and Ω are non-empty with V on $A \times \Omega$. For our purposes here, we shall restrict A to be finite. Let X be a random variable that associates points of Ω to values, x, on the real line and has an underlying distribution P. Thus, our decision problem is to

$$\max_{a \in A} \int V(a,x)\, dP(x) \quad , \tag{1}$$

where V is Lebesque integrable.

The problem is one under risk whenever P is known. More commonly in practice, however, one must deal with decisions under uncertainty where P cannot be specified. The normal procedure for dealing with decision problems under uncertainty is to select a principle of choice, e.g. Maximin, and effectively make assumptions about the probabilities of events over Ω.

2. A MAXIMUM ENTROPY PRINCIPLE.

One way of solving the decision problem under uncertainty is to assume a distribution for X and then solve (1). This procedure is implicit with the Laplace Principle of choice where one assumes the states of nature are equally-likely to occur. In this case the decision maker presumably views nature such that pure random chance is the only vital operant and the only perceived knowledge of P is the range of the random variable X. The fact that different decision makers percieve information pertaining to P differently accounts for the variety of principles that are employed in such analyses. Any process of making assumptions about P should account for all of the known facts and the perception of chance to the decision maker.

Assuming Distributions with Known Moments.

A systematic procedure for assuming distributions that has evolved over the past several years [18] is to derive a distribution that maximizes the entropy, subject to a set of moment constraints that reflect known facts about P. This involves solving the optimization problem

$$\max: \quad H(P) = -\int \log(dP(x)/dx)\, dP(x) \tag{2}$$

$$\text{s.t.} \quad \int dP(x) = 1 \quad , \; dP(x) \geq 0$$

$$\int \xi_k(x) dP(x) = \mu_k \quad , \; k = 1, 2, \ldots$$

where $H(P)$ is the entropy associated with the random variable X and ξ_k is an integrable function of X with μ_k constant for $k = 1, 2, \ldots$. An intuitively appealing interpretation of $H(P)$ is that it is a quantitative index of the amount of information that would be required in order to specify outcomes from Ω with certainty. Solutions to (2) can be obtained using conventional techniques, such as the calculus of variations.

We note that there are at least two constraints, since P is a probability distribution function and hence dP must be nonnegative and sum (or integrate) to 1. It can easily be shown that with these constraints plus knowledge that the range of X is a finite interval of R^1, the maximum entropy distribution is the uniform. Now if X is continuous on $(0, \infty)$ with $\xi_1(x) = x$ and $\xi_j(x) = 0$, for every $j \neq 1$, then the corresponding solution to (2) is the exponential. These and other maximum entropy distributions are given in the Appendix for some cases where prior knowledge about P can be specified through moments of X. We note that this procedure is actually conservative, since it generates distributions that have maximum dispersion. Smith [19] discusses the equivalence of the Maximum Entropy and Minimax Principles under certain conditions.

Assuming Distributions with Known Bounds on Moments.

It is contended that in practice there is usually some knowledge about the states of nature for a decision problem, but it cannot be expressed directly in the form of the constraints in (2). This knowledge rarely extends beyond the first two moments and it is usually in the form of known bounds on particular event probabilities and moments. Let us now restrict ourselves to an X that takes on discrete values, $x_1, x_2, \ldots$ with probabilities $p_1, p_2, \ldots$ respectively. The problem of assuming a distribution over Ω is now stated as

$$\max_P : \quad H(P) = -\sum_i p_i \log p_i$$

$$\text{s.t.} \qquad \sum_i p_i = 1 \quad , \quad p_i \geq 0 \tag{3}$$

$$\alpha_{jk} \leq \sum_{i \in I_k} \xi_k(x_i) p_i \leq \beta_{jk} \quad ; \quad j = 1, 2, \ldots \quad ; \quad k = 0, 1, \ldots \quad ,$$

where I_k is a set of integers with α_{jk} and β_{jk} constants and $\xi_0(x_i) = 1$ for each i. The solution to (3) is not generally obtainable in a simple closed form. It can, however, be solved numerically without difficulty. We observe that (3) is a standard nonlinear programming problem of the form

$$\max_P \quad H(P)$$

$$\text{s.t.} \qquad A_p^T \leq g$$

$$p \geq g$$

where $H(.)$ is concave and separable and the constraints are all linear in p. This problem can be solved by the conventional methods of Lagrangian penalties, reduced gradient, local linearization, or separable programming.

Once (3) is solved and a discrete probability function p is determined then the remaining problem is to select an optimum alternative from the set A. As mentioned previously, the decision problem is now treated as one involving risk. Letting X take on values x_j, $a_i \in A$, and $v_{ij} = V(a_i, x_j)$, $\forall i = 1,\ldots, m; j = 1,\ldots, n$, our discrete analog to (1) is to chose a_i^* such that

$$EV(a_i^*, X) = \max_i \sum_j p_j v_{ij} \tag{4}$$

3. A NUMERICAL EXAMPLE.

We shall demonstrate the GMEP by application to an oil spill abatement planning problem that is of current concern to the U.S. Naval Facilities Engineering Command. Although preventive policies and procedures are enforced, accidental and unavoidable oil spills do occur randomly in both at-sea and in-harbor waterways. It is therefore necessary that clean-up capabilities be provided to remove these spills from the water. A decision was made at the beginning of this decade to improve our capabilities of dealing with these spills. The program development for harbor oil spills, which is of interest here, is to develop systems for removing such spills efficiently and economically. An intermediate effort calls for the utilization of the "best" commercially available equipment and hardware, which are to be refined with Navy-developed techniques for longer range efforts. Decision problems exist in specifying combinations of equipment, policies, and manpower to cope with oil spills occurring in these harbor regions.

Formulation.

The problem of concern is, for a fixed region, to make an optimal selection from among the alternative systems available for cleaning up oil spills. The states of nature are described in Table I according to the environmental conditions, number and volume of spills, and relative confinement of the spilt oil.

Although there are many factors that affect clean-up performance, these are known to be the most pertinent [28]. Here we have simplified our problem by aggregating these conditions and defining only twelve states. For example; the environmental conditions include such factors as wind and current velocity, sea state, air and water temperature, wave height, and many other factors that influence clean-up performance. Moreover, the relative effects of these influences vary from time to time and seasonally. For our purposes here, we are aggregating these factors into "GOOD" and "FAIR/POOR" classes and neglecting any time variations. Similarly, we assume there is a known threshold for the number of spills that separates the frequency classes "FEW" and "MODERATE." Accepted ranges have been established for small, medium, and large volume spills as 0-100, 100-1000, and above 1000 gallons, respectively [28]. This classification applies to both pierside and open area harbor oil spills.

TABLE I

Oil Spill Abatement States

		ENVIRONMENTAL CONDITIONS			
		Good		Fair/Poor	
		SPILL FREQ.		SPILL FREQ.	
CONFINEMENT	VOLUME	Few	Mod.#	Few	Mod.#
Pierside	Small Vol.	1	2	3	3
	Med. Vol.	1	4	5	6
Open Area	Med. Vol.	7	8	9	9
	Large Vol.	7	10	11	12

Every choice of clean-up capability must include the functions of containment, removal, storage/transfer, and separation. Although combinatorially a large number of potential alternative systems result from combinations of the commercially available subsystems, the set can be reduced to a reasonable number. Many of these subsystems have already been evaluated in terms of their potential as alternatives (Widawsky [25]). The following four alternatives typify the ultimate decision problem for a particular region:

a_1 = contract all clean-up activities.

a_2 = contract for pierside clean-up and procure equipment set A for open area spills.

a_3 = procure equipment set B for pierside clean-up and contract for open area spills.

a_4 = procure equipment set C for all spills.

The decision matrix for this problem is given in Table II, which contains the corresponding values for combinations of these 4 alternatives and the 12 states of nature from Table I. Here the matrix entries, v_{ij}, represent average equivalent annual costs.

TABLE II. Value Matrix

A \ Ω	1	2	3	4	5	6	7	8	9	10	11	12
a_1	10	15	18	18	18	21	17	20	32	25	40	50
a_2	10	15	18	18	18	21	20	24	26	24	28	32
a_3	12	8	12	10	10	14	17	20	32	25	40	50
a_4	10	12	15	14	12	14	14	18	28	32	38	42

As previously mentioned, one usually has more information about a decision problem than that given in Table II. For example, in a particular region it may be known that infrequently occurring small and medium volume pierside spills during good environmental conditions are more probable than, say, the remaining 11 states described in Table II. This translates to the constraint: $p_i < p_1$, $\forall\, i = 2, \ldots, 12$. It might further be known from experience that $p_i \leq p_2$, $\forall\, i = 3, \ldots 12$ and that p_1 lies somewhere between 0.25 and 0.60. Let us further typify our numerical example by including this and additional information in the following set of constraints:

$p_i \leq p_1,\ i = 3, \ldots, 12$	$.25 \leq p_1 \leq .60$
$p_1 + p_2 \geq p_3 + p_5 + p_6$	$.25 \leq p_2 \leq .50$
$p_5 \leq p_6$	$.10 \leq p_3 \leq .40$
$p_{10} \leq p_7$	$0 \leq p_i \leq .30,\ i = 4, 11, 12$
$p_9 \leq p_8$	$0 \leq p_i \leq .50,\ i = 5, \ldots, 9$
$p_{12} \leq p_{11} + p_6$	$0 \leq p_{10} \leq .20$
$p_7 \geq p_{10} + p_{11} + p_{12}$	

Solution by GMEP.

A local stepwise gradient descent algorithm of the feasible directions class [2] was used to obtain the solution:

$$p_1^* = .250,\ p_2^* = .250,\ p_3^* = .100,\ p_4^* = .047$$
$$p_5^* = .044,\ p_6^* = .051,\ p_7^* = .082,\ p_8^* = .048$$
$$p_9^* = .047,\ p_{10}^* = .027,\ p_{11}^* = .027,\ p_{12}^* = .027$$

with $H(p^*) = 2.144$. Thus, from (4) we have for the data of Table II.

$$EV(a_i^*, X) = \min\,[17.7,\ 17.0,\ 14.8,\ 15.3] = 14.8$$

Since the v_{ij} values are costs in this case, we must minimize $EV(a_i,X)$. So for this region the appropriate choice of alternative is a_3, contract all open area spills and procure equipment B for pierside spills, based on the ME Principle.

Obviously, there are other relevant factors that have been omitted in this example. The actual selection of harbor oil spill clean-up systems involves many states of nature, some of which are peculiar to given regions. It is also worth noting that in practice the problem of solving (2) or (3) cannot be separated from a statistical estimation problem.

4. OTHER AREAS OF APPLICATION AND FINAL REMARKS.

The GMEP described is a useful method for analyzing decision problems under uncertainty with bounds on event probabilities and moments. For this reason it is particularly amenable to problems in systems design and evaluation whereby knowledge about the future states of nature is sparse and generally quite limited. Such is the case for the oil spill recovery system selection problem in the numerical example. We shall now summarize four conventional areas of operations research within which the GMEP offers potential for systems evaluation.

Location Analysis.

A basic problem in location-allocation theory is to find optimum locations for a set of new facilities with respect to a set of existing ones. One selects an arrangement that results in a minimum cost associated with making transitions among facilities. Although deterministic versions of this problem have been studied extensively, the more realistic stochastic version has received little attention (Weber [3], Cooper [24], Seppala [17]). The transitions among facilities result from demands that are random and are typically assumed to be normally distributed for analytical convenience. We argue that there is usually some knowledge about these demands and that it can be expressed in the form of the constraints in (3). Hence, a reasonable generalization of the location problem is to incorporate maximum entropy demand distributions.

Inventory Systems.

Another area within which system performance and effectiveness are largely affected by randomly occuring demands is inventory. Virtually all inventory decisions are affected by the uncertainties associated with demands. There are numerous models and decision rules available for prescribing such things as the times, amounts, and types of items for order provided the probability structure of the demands is known. Richards and Thomas [15] applied the maximum entropy concept to select an appropriate family of demand distributions to be used in policy level decisions for a large scale multi-item inventory system. In [15] only moment constraints in the form of (2) were considered. Again, more realistic input information can be accommodated for evaluating these and related inventory decisions.

Queueing Systems.

Akin to the inventory problem are queueing problems that arise from arriving customers or items having to await service (see Rosenshine [16] for a recent review). Although queueing theory provides numerous developments for evaluating systems, most of the literature is devoted to ramifications of fixed and known arrival processes, service processes, and operating conditions. For the development and design of new systems, however, one is usually trying to select or prescribe the best choice of service system and related activities for a given arrival process. There is a need for exploring such procedures as the GMEP for this purpose.

Reliability

A fourth area of potential application is in evaluating the reliability of operating systems. While reliability theory for the most part has evolved about hardware components and systems, there is current interest in adopting similar quantitative techniques for human error evaluation. It is recognized that most systems warranting analysis are man-machine systems and the human aspects of the problem cannot be overlooked. Unfortunately, the human data tends to be more variable and sensitive to environmental conditions. Regulinski [14] has proposed using maximum entropy concepts to deal with this problem. A similar and related problem area is the relatively new "software reliability" area.

Indeed these four areas of application are not exhaustive. Many problems arise in the analysis and evaluation of systems that require descriptions of random phenomena and hence one or more probability distributions must be assumed. The amount of information available for supporting such assumptions is generally quite limited. Obviously, to some decision makers and for certain applications the rationality of the ME Principle might be questionable. For cases where this rationality is acceptable, however, the generalization described here is useful for making the best utilization from available known facts in analyzing decisions under uncertainty.

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APPENDIX

ME DISTRIBUTIONS WITH KNOWN MOMENTS

The formulation of the problem of determining the ME distribution with known moment constraints is given in (2). Although this optimization problem may be solved for either cases of continuous or discrete X, the results follow directly by inspection. It is known (see Kullback [11]) that for an exponential family of probability functions

$$f(x|\theta) = c(\theta)\, h(x) \exp[\sum_i \pi_i(\theta)\, T_i(x)] ,$$

where $\theta \epsilon \Theta$, the set of parameters, and

$$c(\theta)^{-1} = \sum_x h(x) \exp[\sum_i \pi_i(\theta)\, T_i(x)]$$

for discrete X (and the obvious analog for the absolutely continuous case), is a maximum entropy distribution whenever the constraints in (2) are of the form

$$E[T_i(x)] = \mu_i .$$

Thus one can derive the ME distribution by merely factoring the pdf or dpf as the case may be. Table 1 gives some classical ME pdf's and associated constraints. Some discrete probability functions are given in Table 2.

TABLE 1. Some Classical Maximum Entropy Probability Density Functions

Distribution	Conditions	f(x)	mean	Variance
Uniform	$x \in (a,b)$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\xi_1(x) = x$ $x \in (0,\infty)$	$\lambda e^{-\lambda x}$, $\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Laplace	$\xi_1(x) = \lvert x \rvert$ $x \in (-\infty,\infty)$	$\frac{e^{-\left\lvert\frac{x-\alpha}{\beta}\right\rvert}}{2\beta}$, $-\infty < \alpha < \infty$ $0 < \beta < \infty$	α	$2\beta^2$
Error Function	$\xi_1(x) = x^2$ $x \in (-\infty,\infty)$	$\frac{he^{-h^2x^2}}{\sqrt{\pi}}$, $0 < h < \infty$	0	$\frac{1}{2h^2}$
Normal	$\xi_1(x) = x$ $\xi_2(x) = x^2$ $x \in (-\infty,\infty)$	$\frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$, $\sigma > 0$	μ	σ^2

TABLE 1. (Cont'd.)

Distribution	Conditions	f(x)	mean	Variance
Pareto	$\xi_1(x) = \ln x$ $x \in (\gamma,\infty)$ $\gamma > 0$	$\alpha\gamma^{\alpha} x^{-(\alpha+1)}$, $\alpha > 0$	$\frac{\alpha\gamma}{\alpha-1}$	$\frac{\alpha\gamma^2}{(\alpha-1)^2(\alpha-2)}$
Gamma	$\xi_1(x) = x$ $\xi_2(x) = \ln x$ $x \in (0,\infty)$	$\frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\ \beta^{\alpha}}$, $\alpha > 0$, $\beta > 0$	$\alpha\beta$	$\alpha\beta^2$
Rayleigh	$\xi_1(x) = \ln x$ $\xi_2(x) = x^2$ $x \in (0,\infty)$	$\frac{xe^{-\frac{1}{2}(x/\alpha)^2}}{\alpha^2}$, $\alpha > 0$	$\alpha\sqrt{\frac{\pi}{2}}$	$2\alpha^2(1 - \frac{\pi}{4})$
Maxwell	$\xi_1(x) = \ln x^2$ $\xi_2(x) = x^2$ $x \in (0,\infty)$	$\frac{2x^2\ e^{-\frac{1}{2}(x/\alpha)^2}}{\alpha^3\sqrt{2\pi}}$	$\frac{4\alpha}{\sqrt{2\pi}}$	$3\alpha^2(1 - \frac{8}{3\pi})$
Beta	$\xi_1(x) = \ln x$ $\xi_2(x) = \ln(1-x)$ $x \in (0,1)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $\alpha > 0$, $\beta > 0$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

TABLE 2. Some Classical Maximum Entropy Discrete Probability Functions.

Distribution	Conditions	p(x)	mean	Variance
Uniform	$x = a, a+1, \ldots, a+r$	$\frac{1}{r}$, $r = 1,2,\ldots,$ $-\infty < \alpha < \infty$	$\frac{2a+r}{2}$	$\frac{r^2}{12}$
Bernoulli	$\xi_1(x) = x$ $x = 0,1$	$p^x(1-p)^{1-x}$, $0 < p < 1$	p	$p(1-p)$
Geometric	$\xi_1(x) = x$ $x = 0,1,\ldots$	$p(1-p)^x$, $0 < p < 1$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
Binomial	$\xi_1(x) = x$ $\xi_2(x) = \ln[x!(n-x)!]$ $x = 0,1,\ldots,n$	$\binom{n}{x} p^n(1-p)^{n-x}$, $0 < p < 1$ $n = 1,2,\ldots$	np	$np(1-p)$
Negative Binomial	$\xi_1(x) = x$ $\xi_2(x) = \ln[(r+x-1)!]$ $- \ln[x!]$ $x = 0,1,\ldots$	$\binom{r+x-1}{x} p^r(1-p)^x$, $0 < p < 1$ $r = 1,2,\ldots$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$

TABLE 2. (Cont'd.)

Distribution	Conditions	p(x)	mean	Variance
Poisson	$\xi_1(x) = x$ $\xi_2(x) = \ln x!$ $x = 0,1,\ldots$	$\frac{\lambda^x e^{-\lambda x}}{x!}$, $\lambda > 0$	λ	λ
Hypergeometric	$\xi_1(x) = \ln[(Np-x)!x!] + \ln[(Nq-n-x)!(n-x)!]$ $x = 0,1,\ldots,n$	$\frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$, $p = 0, {}^1/_N, {}^2/_N, \ldots, 1$ $q = 1 - p$ $N = 1,2,\ldots$ $n = 1,\ldots,N$	np	$npq\left(\frac{N-n}{N-1}\right)$
Zeta	$\xi_1(x) = \ln x$ $x = 1,2,\ldots$	$\frac{x^{-(\alpha+1)}}{\zeta(\alpha+1)}$, $\alpha > 0$, $\zeta(\cdot) \equiv$ Riemann Zeta Function	$\frac{\zeta(\alpha)}{\zeta(\alpha+1)}$	$\frac{\zeta(\alpha-1)\zeta(\alpha+1)-[\zeta(\alpha)]^2}{[\zeta(\alpha+1)]^2}$

REFERENCES

1. Abdel-Khalik, A. R., "The Entropy Law, Accounting Data, and Relevance to Decision-Making", The Acct. Rev., 49, 271-283 (1974).

2. Clasen, R. J. Graves, G. W., and Lu, J. Y., "Sortie Allocation by a Non-linear Programming Model for Determining a Munitions Mix," Rand Report R-1411DDPAE, March, 1974.

3. Cooper, L., "A Random Locational Equilibrium Problem," J. Regional Sci., 14, 47-54 (1974).

4. Cozzolino, J. M. and Zahner, M. J., "The Maximum-Entropy Distribution of the Future Market Price of a Stock", Opns. Res., 23, 1200-1211 (1973).

5. Duncan, G. T., "Optimal Diagnostic Questionnaires", Opns. Res., 23, 22-32 (1975).

6. Gokhale, D. V., "Approximating Discrete Distributions, with Applications", J. Am. Statis, Assoc., 68, 1009-1012 (1973).

7. Griffeath, D. S., "Computer Solution of the Discrete Maximum Entropy Problem", Technometrics, 14, 891-897 (1972).

8. Herniter, J. D., "An Entropy Model of Brand Purchase Behavior", J. Mktg. Res., 10, 361-375 (1973).

9. Horowitz, A. R. and Horowitz, I., Entropy, Markov Processes, and Competition in the Brewing Industry", J. Ind. Econ., 16, 196-211 (1968).

10. Horowitz, I., Employment Concentration in the Common Market: An Entropy Approach", J. Royal Stat. Soc., 133, 463-479 (1970).

11. Kullback, S., Information Theory and Statistics, Dover Publications, New York, 1968.

12. Lee, R., "A Markovian Entropy-Maximizing Model of Population Distribution, Environ. and Planning A, 6, 693-702 (1974).

13. Pollock, S. M., "Search Detection and Subsequent Action: Some Problems on the Interfaces, Opns. Res., 19, 559-586 (1971).

14. Regulinski, T. L., "Elements of Information Theoretic Methods: Tutorial Introduction", IEEE Trans. on Reliab., R-18, 82-87 (1969).

15. Richards, F. R. and Thomas, M. U., "A Rationale for Selecting a Family of Demand Distributions for Inventory Systems", Proc. AIIE Spring Conf., pp. 269-273, American Institute of Industrial Engineers, 1974.

16. Rosenshine, M., "Queueing Theory: The State-of-the-Art", AIIE Trans., 7, 257-267 (1975).

17. Seppala, Y., "On a Stochastic Multi-Facility Location Problem", AIIE Trans., 7, 56-62 (1975).

18. Shannon, C. E. and Weaver, W., The Mathematical Theory of Communications, University of Illinois Press, Urbana, 1969.

19. Smith, S. A., "A Derivation of Entropy and the Maximum Entropy Criterion in the context of Decision Problems", IEEE Trans. on Sys., Man, and Cybern., SMC-4, 157-163 (1974).

20. Tapiero, C. S. and Lewin, A. Y., "The Concept and Measurement of Centrality - An Information Approach:, Decision Sci., 4, 314-328 (1973).

21. Theil, H., "On the Use of Information Theory Concepts in the Analysis of Financial Statements", Management Sci., 15, 459-480 (1969).

22. Thomas, M. U., "A Human Response Model of a Combined Manual and Decision Task", IEEE Trans. on Sys., Man, and Cybern., SMC-3, 478-484 (1973).

23. Tribus, M., Rational Descriptions, Decisions, and Designs, Elmsford, N.Y.: Pergamon, 1969.

24. Webber, M. J., Impact of Uncertainty on Location, MIT Press, 1972.

25. Widawsky, A., "Development of Harbor Oil Spill Removal - Recovery Systems: Phase I", Proc. of Joint Conf. on Prev. and Contr. of Oil Spills, pp. 379-386, American Petroleum Institute, 1975.

26. Wilson, A. G., "Further Developments of Entropy Maximizing Transport Models", Transpn. Plan. Tech., 1, 183 (1973).

27. Wilson, C. J. and Philippatos, G. C., "Entropy, Market Risk, and the Selection of Efficient Portfolios:, Appl. Econ., 4, 209-220 (1972).

28. Wilson, J. E., Nadeau, P. F., and Premack, J. S., "Navy Harbor Oil Spill Cleanup: A Progress Report", Proc. of Joint Conf. on Prev. and Contr. of Oil Spills, pp. 373-378, American Petroleum Institute, 1975.

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